W1/1 Rings (Recap)

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Commutative Algebra

1. Recap: Rings & Modules

1.1 Rings

Ring: $(A,+,\cdot)$ s.t. $(A,+)$ is an abelian group (with 0)
$(A,\cdot)$ is a monoid (with 1)

and distributivity holds: $a(b+c) = ab + ac$
$(b+c)a = ba + ca$

So: always associative, unital (with 1)

In this course (& CA), rings are ALWAYS COMMUTATIVE (unless specified otherwise)

Exm: • $\mathbb{Z}$, $\mathbb{Z}[i] = \{a+bi : a,b \in \mathbb{Z}\} \subseteq \mathbb{C}$, $\underline{0} = A = \{0\}$ zero ring/null ring

• Field K

• If A is a ring, polynomial rings: $A[x]$, $A[x_1,\dots,x_n]$, $A[\underline{X}]$ with $\underline{X}$ a set of indeterminates

• If M is an abelian group, $\mathrm{End}(M) := \{f: M \to M \mid f \text{ group hom}\}$ is a (usually) noncommutative ring.
Similarly: K field, V_K K-vector space of dim d
$\Rightarrow \mathrm{End}(V_K) \cong M_d(K)$ is nc if $d > 1$

• X top. space, $C(X) := \{f: X \to \mathbb{R} \mid f \text{ continuous}\}$ with pointwise operations is a ring

• If $(A_i)_{i\in I}$ is a family of rings, $\prod_{i\in I} A_i$ is a ring

• (Formal) power series: A ring, $A[[x]]$ consists of sequences

- (Formal) power series: A ring, $A[\![x]\!]$ consists of sequences $f: \mathbb{N}_0 \to A$ with
$$(f+g)(n) := f(n) + g(n)$$
$$(fg)(n) = \sum_{k=0}^{n} f(k) f(n-k) \qquad \text{(Cauchy product)}$$

<u>Notation:</u> $f \leadsto \sum_{n=0}^{\infty} f(n) x^n$

- Other products are possible on sequences:

<u>E.g.</u> – pointwise product: $(fg)(n) = f(n) g(n)$
– for $f, g: \mathbb{N} \to A$, Dirichlet convolution:
$$(f * g)(n) := \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right)$$

- $(A_i)_{i \in I} \subseteq A$ family of subrings $\Rightarrow \bigcap_{i \in I} A_i$ is a ring.

<u>Notions:</u> $a \in A$ is

invertible (= a unit) $\Leftrightarrow \exists b \in A: \ ab = 1$

zero-divisor $\Leftrightarrow \exists b \in A \setminus \{0\}: \ ab = 0$

nilpotent $\Leftrightarrow \exists n \geq 1: \ a^n = 0.$

A^* $(= A^\times, \mathcal{U}(A))$ ... group of invertible elements

$A^\bullet$ ... non-zero-divisors (non-standard notation) rare, but used in Ferretti-book

A is a domain (integral domain, integral)

$\Leftrightarrow A \neq \underline{0}$ and $ab = 0 \Rightarrow a = 0$ or $b = 0$

$\Leftrightarrow 0$ is the only zero-divisor in A

<u>Exm:</u> $f = \sum_{n=0}^{\infty} f(n) x^n \in A[\![x]\!]$ is invertible $\Leftrightarrow f(0) \in A^\times$

[Exercise: „$\Rightarrow$" is „obvious", „$\Leftarrow$" Write down eq's for $fg = 1$]

$n=0$

[Exercise: „$\Rightarrow$" is „obvious", „$\Leftarrow$" Write down eqn's for $fg=1$]

If $A \subseteq B$ are rings (with A a subring of B), $S \subseteq B$ a subset

$$\Rightarrow \quad A[S] := \bigcap_{\substack{A \subseteq A' \subseteq B \text{ subring} \\ S \subseteq A'}} A'$$

is the subring of B obtained by adjoining S to A.

$$A[S] = \left\{ \sum_{i=1}^{m} a_i s_i \;,\; a_i \in A,\, s_i \in S \right\}$$

<u>IDEALS:</u> $I \subseteq A$ is an ideal if $\emptyset \neq I$, $I + I \subseteq I$, and $AI \subseteq I$.

(Notation: $I \trianglelefteq A$)

<u>Quotient:</u> $A/_I = \{ a + I : a \in A \}$ is then a ring.

(other notations: $a + I = \bar{a} = [a] = [a]_I$)

$a + I = b + I \Leftrightarrow a - b \in I \Leftrightarrow a \equiv b \mod I$

If $X \subseteq A$, then (X) (or $\langle X \rangle$, $\langle X \rangle_A$) is the ideal generated by X, i.e., the smallest ideal containing X.

$$(X) = \sum_{x \in X} Ax \;, \quad \text{e.g. } (a) = Aa, \quad (a,b) = Aa + Ab$$

$\llcorner$ principal ideal

<u>Exm:</u> •) Ideals of $\mathbb{Z}$ are $\{ d\mathbb{Z} : d \in \mathbb{N}_0 \}$

•) $(x,y) \trianglelefteq K[x,y]$ is not principal (Check!)

•) $I = A \Leftrightarrow 1 \in I \Leftrightarrow I \cap A^\times \neq \emptyset$.

<u>Def:</u> $I \trianglelefteq A$ is

1) a prime ideal if I is proper ($I \subsetneq A$) and
$\forall a, b \in A: \; ab \in I \rightarrow a \in I$ or $b \in I$

$\mathrm{Spec}(A) := \{ P \ntrianglelefteq A : P \text{ prime} \}$

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2) a **maximal ideal** if $I \subsetneq A$ and

$\forall J \trianglelefteq A : I \subsetneq J \Rightarrow J = A$

$\mathrm{Max}(A) := \{ M \trianglelefteq A : M \text{ maximal} \}$

Thm: $I \trianglelefteq A$ prime $\Leftrightarrow$ A/I domain

$I \trianglelefteq A$ max. $\Leftrightarrow$ A/I field

So max $\Rightarrow$ prime. Every $I \subsetneq A$ is contained in a maximal ideal (using Zorn's lemma).

Exm: •) prime ideals of $\mathbb{Z}$: $\{ p\mathbb{Z} : p \text{ prime number} \} \cup \{ \underline{0} \}$

Each $p\mathbb{Z}$ is maximal ($\mathbb{Z}/p\mathbb{Z}$ is a field).

•) In $K[x,y]$ (K field) $\quad \underline{0} \subseteq (x) \subseteq (x,y)$ are prime. Of these, only (x,y) is maximal

•) $\underline{0}$ is prime $\Leftrightarrow$ A domain

IDEAL CONSTRUCTIONS:

$I + J = \{ a + b : a \in I, b \in J \} = (I, J)$

$I \cap J \supseteq I \cdot J = (\{ ab : a \in I, b \in J \}) = \left\{ \sum_{i=1}^{m} a_i b_i \mid a_i \in I, b_i \in J \right\}$

radical: $\sqrt{I} := \{ a \in A \mid \exists n \geq 1 : a^n \in I \}$ is an ideal

[non-trivial: $a, b \in \sqrt{I} \Rightarrow a + b \in \sqrt{I}$. Say wlog $a^n, b^n \in I$

$$\Rightarrow (a+b)^{2n} \overset{\text{commutativity}}{=} \sum_{i=0}^{2n} \binom{2n}{i} \underbrace{a^i b^{2n-i}}_{\begin{cases} a^i \in I \text{ if } i \geq n \\ b^{2n-i} \in I \text{ if } i \leq n \end{cases}} \in I$$

]

Def: (1) $\mathcal{N}(A) := \sqrt{\underline{0}} = \{a \in A : a \text{ nilpotent}\}$ is the **nilradical** of A

(2) The **Jacobson radical** $\mathcal{J}(A)$ (or $J(A)$) is the intersection of all maximal ideals.

If $A \neq \underline{0}$, $\mathcal{N}(A)$ and $\mathcal{J}(A)$ are proper!

Lemma 1.1 (1) $\mathcal{N}(A) \subseteq \mathcal{J}(A)$

(2) $\mathcal{J}(A) = \{a \in A \mid \forall b \in A : 1 - ab \in A^\times\}$

Proof: (1) Let $a \in \mathcal{N}(A)$, $n \geq 1$: $a^n = 0$

Let $M \in \operatorname{Max}(A)$, $a^n \in M \xRightarrow{M \text{ prime}} a \in M$

$\Rightarrow a \in \bigcap_{M \in \operatorname{Max}(A)} M = \mathcal{J}(A)$.

(2) „$\subseteq$" Let $a \in \mathcal{J}(A)$, let $b \in A$.

For all $M \in \operatorname{Max}(A)$:

$1 - \underbrace{ab}_{\in M} \notin M$ (since $1 \notin M$)

$\Rightarrow (1 - ab) = A \Rightarrow 1 - ab \in A^\times$

„$\supseteq$" Let $a \in A$ s.t. $\forall b \in A$: $1 - ba \in A^\times$

Let $M \in \operatorname{Max}(A)$. Suppose $a \notin M$

$\Rightarrow A = (a, M) \Rightarrow \exists x \in A, m \in M : 1 = ax + m$

$\Rightarrow \underbrace{1 + (-x)a}_{= m} \in M$ ↯ $1 + (-x)a \in A^\times$

$\square$

Lemma 1.2

(1) **(Prime Avoidance)** Let $I \trianglelefteq A$, $P_1, \dots, P_n \in \operatorname{Spec}(A)$

If $I \subseteq P_1 \cup \dots \cup P_n \Rightarrow \exists k : I \subseteq P_k$.

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(2) Let $I_1, \dots, I_n \trianglelefteq A$, $P \in \mathrm{Spec}(A)$

If $I_1 \cap \dots \cap I_n \subseteq P \Rightarrow \exists k : I_k \subseteq P$.

<u>Proof:</u> (1) Induction on n. <u>$n=1$</u> ✓ <u>$n-1 \to n$</u>.

By IH, wlog $\forall i \; \exists a_i \in I \setminus \bigcup_{j \neq i} P_j$.

$\Rightarrow a_i \in P_i$

hat means omitted

$$a := \sum_{j=1}^{n} a_1 \cdots \hat{a}_j \cdots a_n = \underbrace{a_1 \cdots \hat{a}_i \cdots a_n}_{\notin P_i} + \underbrace{\sum_{j \neq i} a_1 \cdots \hat{a}_j \cdots a_n}_{\in P_i} \quad (\forall i)$$

because otherwise $\exists j \neq i$ s.t. $a_j \in P_i$ by primality

$\Rightarrow \forall i: a \notin P_i$ ↯ $a \in I \subseteq P_1 \cup \dots \cup P_n$

(2) Suppose $\forall j: I_j \not\subseteq P$. Let $a_j \in I_j \setminus P$

$\Rightarrow a_1 \cdots a_n \in I_1 \cdots I_n \subseteq I_1 \cap \dots \cap I_n \subseteq P \xRightarrow{P \text{ prime}} \exists j: a_j \in P$ ↯

□

<u>RING HOMOMORPHISMS</u> $f: A \to B$ ring hom $\Rightarrow \ker(f) \trianglelefteq A$.

$I \trianglelefteq B \Rightarrow f^{-1}(I) \trianglelefteq A$

$P \in \mathrm{Spec}(B) \Rightarrow f^{-1}(P) \in \mathrm{Spec}(A)$ (Check!)

Universal Property (UP) of Quotients:

Let $I \trianglelefteq A$, $\pi: A \to A/I$, $a \mapsto a + I$ canonical epi

For every ring hom $f: A \to B$ with $I \subseteq \ker(f)$, there exists a <u>unique</u> ring hom. $\hat{f}: A/I \to B$ s.t. $f = \hat{f} \circ \pi$

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[dr, "f"'] & A/I \arrow[d, dotted, "\exists! \hat{f}"] \\ & B \end{tikzcd}$$

$$f \searrow \quad \swarrow \exists!\, \hat{f}$$
$$B$$

This UP characterizes $(A/I, \pi)$, $\ker(\hat{f}) = \ker(f)/I$.

<u>Consequence:</u> If $f: A \to B$ is ring hom,

$$A/\ker(f) \cong f(A)$$

<u>Isomorphism Theorems:</u> $I \trianglelefteq A$

(1) $\{ J \trianglelefteq A : I \subseteq J \subseteq A \} \xleftrightarrow{\text{bij}}$ ideals of A/I

$\qquad J \mapsto J/I = \{ a + I : a \in J \}$

$\qquad \pi^{-1}(\bar{J}) = \{ a \in A : \pi(a) \in \bar{J} \} \mapsfrom \bar{J}$

(2) $J \trianglelefteq A$, $I \subseteq J \Rightarrow A/J \cong (A/I)/(J/I)$

(3) $B \subseteq A$ subring,

$\Rightarrow B + I$ subring, $B \cap I \trianglelefteq B$

$$(B+I)/I \cong B/(B \cap I)$$